Bernard Parzysz

Université d'Orléans (France)
Laboratoire André-Revuz
Université Paris-Diderot
22 avenue du Général Leclerc
92260 Fontenay-aux-Roses
FRANCE
parzysz.bernard@wanadoo.fr

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Research

From One Polygon to Another: A Distinctive Feature of Some Ottoman Minarets

Abstract. This article is a geometer's reflection on a specificity presented by some Turkish minarets erected during the Ottoman period which gives them a recognisable appearance. The intermediate zone (*pabuç*) between the shaft and the base of these minarets, which has both a functional and an aesthetic function, is also an answer to the problem of connecting two prismatic solids having an unequal number of lateral sides (namely, two different multiples of 4). In the present case this connection was achieved by creating a polyhedron, the lateral sides of which are triangles placed head to tail. The multiple variables of the problem allowed the Ottoman architects to produce various solutions.

1 Introduction

Each religion developed specific places for worship: synagogues, temples, churches, mosques, and so forth, and the rites and ceremonies for which they were designed conditioned their architecture. Mosques are associated with Islam, and are most certainly perceived throughout the world as its most typical monuments. They accompanied very early the history of this religion, but one of their most emblematic features, the minaret, did not appear immediately and when they did, specificities occurred depending on the period and geographic and cultural area they were built in. As Auguste Choisy writes in his *Histoire de l'architecture*: "Minarets have their own geography, just as steeples have theirs". He then defines "the cylindrical type which is specific of Persia", points out that Egyptian minarets are in the shape of a "shaft with numerous balconies sticking out, its shape being no longer cylindrical but polygonal and its plan changing from one floor to another" and that "the square tower type … is that of Tunisia, Algeria, Morocco" [Choisy 1964: II, 102, my trans.].

As a maths academic I have been developing for some years a particular interest in architecture, and especially in the ways used by architects of all times and places to link up various solids. In this article I want to focus, from the standpoint of a Western geometer and not as an historian of architecture, on a specificity shown by minarets belonging to a given geographic area (Turkey) and period (the Ottoman Empire[1]). The problem is how to connect two different volumes (as for instance a prism and a cylinder) together, aiming at producing a shape as harmonious as possible for the intermediate volume, i.e., other than just by superposing them. It is a very general and classical question for architects, which makes it necessary to bring knowledge of spatial geometry into play, especially the study of intersections of solids. It is this geometrical aspect that I would like to develop in the particular case of Ottoman minarets, because I think it gives a simple and elegant solution to the problem.

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2 Some general points about minarets

A minaret is a tower associated with a mosque, from the top of which a man (*muezzin*) calls the faithful to prayers (*adhan*). The first minarets appeared during the eighth century, nearly a century after Muhammad's death in 632: "The minaret was absent in the early mosques, and its addition was inspired by religious buildings of other religions. The main influence probably came from the churches of Syria" [Kjeilen 2010]. (On the origin of minarets, see also [Kahlaoui 2009]).

Though minarets are widely spread throughout the Islamic world, in some regions mosques generally have no minarets: "In parts of Iran, East Africa, Arabia and much of the Far East many mosques were built without them. In such places the call to prayer is either made from the courtyard of the mosque or from the roof" [Petersen 1996: headword 'minaret'].

In the greater part of the Islamic world, minarets, like mausoleums, are frequently composed of a cylindrical or prismatic shaft standing on a square base, and the connection between them may consist of a mere superimposition, as for instance in Kunya Urgench (Turkmenistan) [Chmelnitsky 2008: 71] (fig. 1).

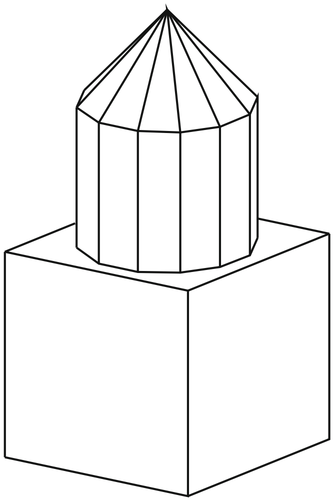

Fig. 1. Kunya Urgench. Schematic bird's eye view of Fakhr al-Din mausoleum (end of twelfth century)

But sometimes, and relatively early, the connection of an octagonal shaft with a square base is made by cutting down the protruding trihedra, as for instance in Cairo mosques: Al-Azhar, 970 [Jacobi 1998: 72], Sultan Hassan, ca. 1360 [Meinecke-Berg 2008: 188] and Sultan Qaytbay, 1472 [Meinecke-Berg 2008: 189]. This happens in Turkey as well: the mosques of Yivli Minare (Antalya, 1230) and Ala-ed-Din (Nigde, 1223) (fig. 2).

This shape can be considered [Volwahsen 1971: 179] as resulting from the intersection of a right-angled prism and a regular pyramid, both of which have a square base (fig. 3).

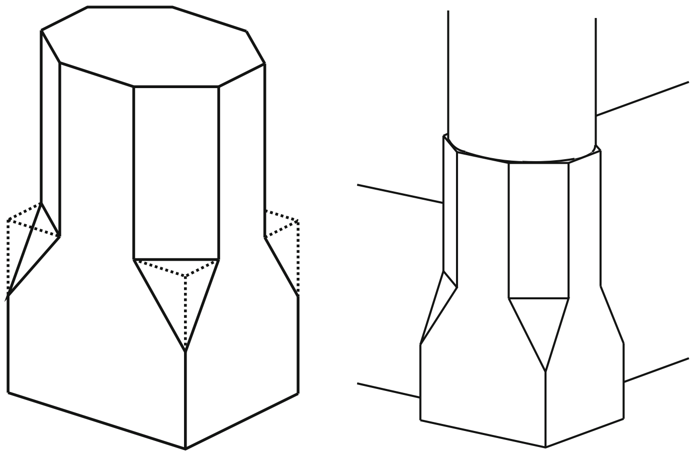

Fig. 2. a, left) Connecting an octagon to a square by cutting down trihedra; b, right) Lower part of the minaret of Ala-ed-Din mosque (Nigde, 1223)

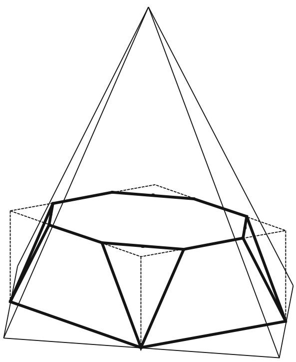

Fig. 3. Intersection of a regular prism and pyramid with square bases

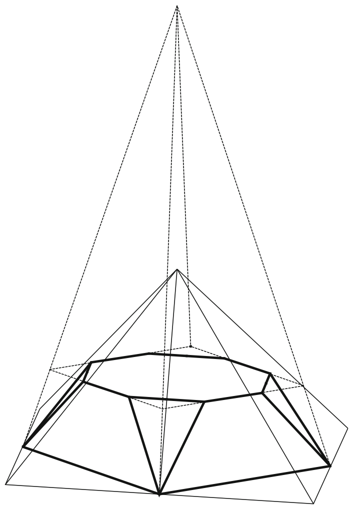

Fig. 4. Intersection of two regular pyramids with square bases

N.B. If the octagon is inscribed in a square smaller than the base, the prism becomes a pyramid and the problem is then that of the intersection of two pyramids with square bases having the same axis (fig. 4).

Let us also note that with a dodecagonal upper part, as for instance at the Döner Kümbet[2] in Kayseri (1276), the cutting down of trihedra will be double (fig. 5).

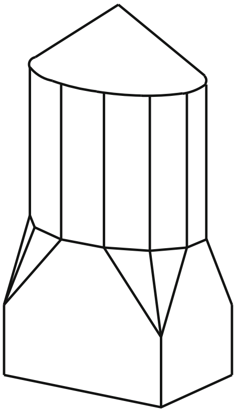

Fig. 5. Kayseri. Schematic bird's eye view of Döner Kümbet: double cutting down of trihedra

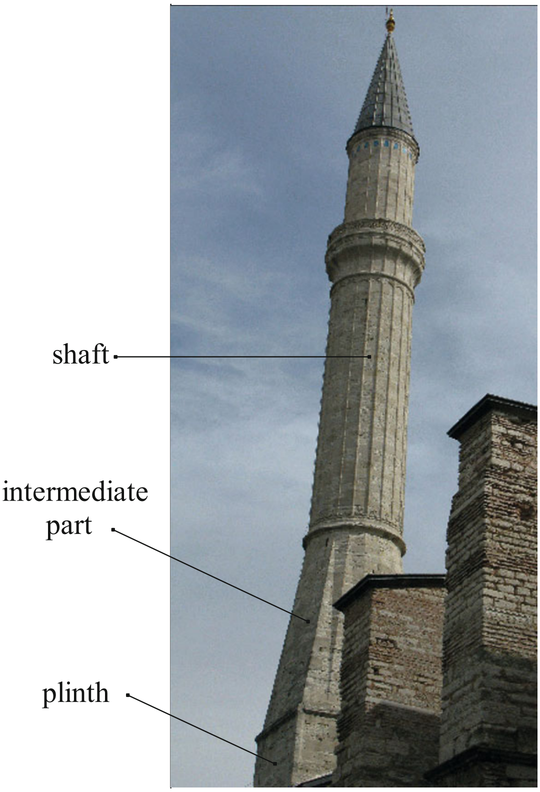

Fig. 6. Istanbul, Aghia Sofya Cami. North-west minaret

As stated by many authors, in Ottoman Turkey minarets frequently take the following form: a very slender cylindrical or prismatic shaft ending in a point and lying on a wider prismatic plinth, to which it is connected by a polyhedral intermediate part (fig. 6):

> The pencil-shaped Ottoman minaret is a tall, faceted or fluted mostly polygonal or cylindrical shaft which has a ring on upper or lower part, resting on triangular buttresses (transition zone) "pabuç" above the base "kürsü". Transition zones mostly decorated with Turkish triangle motives"[3] [Altuğ 2010].

The Turkish triangle mentioned by Altuğ is "a transformation of the curved space of the traditional pendentive into a fanlike set of long and narrow triangles built at an angle from each other" (Britannica online Encyclopaedia). Like pendentives and muqarnas, the main function of Turkish triangles is to realize the junction between the polygonal plan of a building and the circular cupola topping it.

The specific shape of the minaret is possibly inherited from Egyptian Mameluke architecture, which developed in Egypt and Syria between 1250 and 1517:

> In post-Fatimid Egypt minarets developed into a complex and distinctive form. Each tower is composed of three distinct zones: a square section at the bottom, an octagonal middle section and a dome on the top. The zone of transition between each section is covered with a band of muqarnas[3] decoration [Petersen 1996: headword 'minaret'].

> In Mameluke Egypt minarets are divided into three distinct zones: a square section at the bottom, an octagonal middle part and an upper cylindrical part topped with a small cupola. … The transition between two parts is made with a strip of muqarnas. … During the Ottoman era, octagonal and cylindrical minarets replaced square towers. They are often high tapering minarets and, although mosques generally have a single minaret, in big towns they may have two, four or even six of them [Binous et al. 2002: 27-28, my trans.].

3 Study of a specific form of pabuç

3.1 Posing the problem

Translating Altuğ's definition into mathematical terms and observing that the lower section of Ottoman minarets is not always square (see, for instance, §§ 3.2 and 3.5 below), we can say that the base of the prismatic plinth is a convex regular polygon with $4n$ sides (n being an integer number) and the shaft, prismatic as well,[4] is a convex regular polygon with $4kn$ sides (k being an integer number, $k \geq 2$).

The study, based on examples, of a specific – though seemingly common – form of the intermediate part of Ottoman minarets, is the subject of this article. This feature is particularly noteworthy, because it poses, and gives a simple and nice solution to, the following space geometry problem (fig. 7):

> In two horizontal planes let two convex regular polygons, N and P (P being on top of N; for obvious reasons, the diameter of the circle drawn round N is bigger than the diameter of the circle drawn round P), with the same vertical axis and respectively $4n$ and $4kn$ sides ($k \geq 2$). Construct a polyhedron having the same axis, P and N being respectively its upper and lower sides.

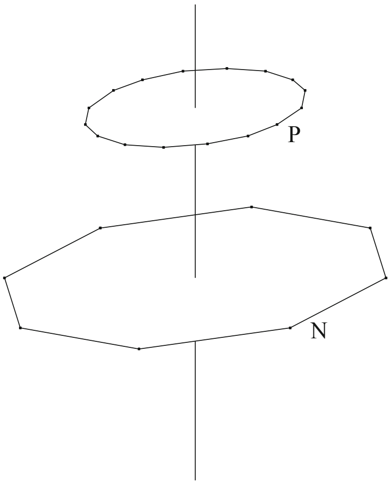

Fig. 7. ($n = 2$, $k = 2$)

The way in which this problem is usually solved in Turkey consists in determining plane polygons – especially triangles, which are *necessarily* plane –, some vertices of which are situated in N and the others in P. The variety of the solutions taken up by Turkish architects for determining such polyhedra will now be illustrated with some examples. (Some of the minarets studied here have been restored at various times, but nevertheless it can be assumed that the repairing did not alter their shape noticeably.) (N.B. Since actually no actual measurements were available, this study is limited to qualitative aspects.)

3.2. Example 1: Minaret of Üftade mosque, Bursa

Fig. 8. Bursa. Üftade Cami

Mehmet Muhyiddin Üftade (Bursa, 1490-1580), was one of the greatest masters of Sufism. In the Minaret of Üftade mosque in Bursa of 1579 (fig. 8) we have $n = 2$ and $k = 2$. Eight sides of the hexadecagon are parallel to a side of the lower octagon, thus creating eight isosceles trapezoids. Therefore the intermediate polyhedron, which is convex, has 16 lateral sides: eight trapezoids separated by eight triangles (in the photographs the edges of the lateral sides of the intermediate polyhedron are emphasised, in order to make them more visible) (fig. 9).

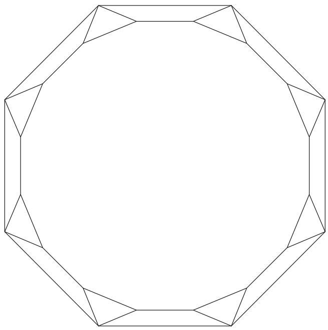

Fig. 9. Bursa, Üftade mosque. Polyhedron seen from above

This arrangement may be seen as the intersection of two regular pyramids with octagonal bases having the same axis (fig. 10).

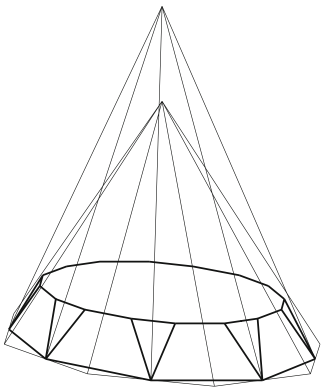

Fig. 10. Intersection of two regular pyramids with octagonal bases

But fig. 8 shows that the arrangement of bricks visually subdivides each trapezoid into three triangles which look isosceles (fig. 11).

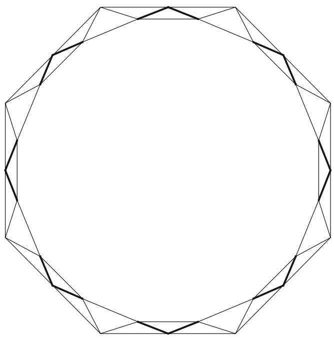

Fig. 11. Bursa, Üftade mosque. Subdivision of the trapezoids

If this is really the case, the ratio of the radii of the circles drawn around the two polygons is fixed. Actually, the length of the sides of a regular convex octagon inscribed in a circle with radius R is $A = 2R \cdot \sin(\pi/8)$ and the length of the sides of a regular convex hexadecagon inscribed in a circle with radius r is $a = 2r \cdot \sin(\pi/16)$. With the above hypothesis we have $A = 2a$. Then $R \cdot \sin(\pi/8) = 2r \cdot \sin(\pi/16)$, or $r = R \cdot \cos(\pi/16) \approx 0.98R$. The two radii are almost equal, which obviously is not the case in fig. 8. A consequence is that, among the three triangles subdividing a trapezoid, only one is isosceles.

Why the trapezoids are subdivided in such a way is also a question. I am inclined to favour a concern with aesthetics, aiming at unifying the visual aspect of the polyhedron: eventually, what is given to see is a frieze of (almost) equal triangles arranged head to tail (this is only a convenient term to indicate that the horizontal sides of two adjacent triangles are situated, one on the upper polygon, and the other on the lower polygon). This is in fact a specific use of Turkish triangles.

3.3. Example 2: North-west minaret of Aghia Sofya mosque, Istanbul

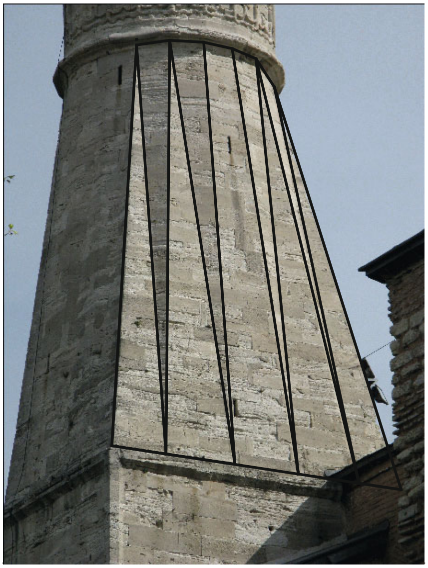

Fig. 12. Istanbul, Aghia Sofya Cami. North-west minaret

The Aghia Sofya mosque includes four minarets. The two western minarets (second half of the sixteenth century, fig. 12) are the work of the famous architect Mimar Sinan (1491-1588), who built almost two hundred buildings in Istanbul. He also built the Selimiye mosque in Edirne (1574), considered his masterpiece. Having four minarets is not a rare feature in Turkey, and the reason generally put forward is that mosques with several minarets were those built by a sultan: "In the major cities of the [Ottoman] empire mosques were built with two, four or even six minarets. At some point it seems to have been established that only a reigning sultan could erect more than one minaret per mosque" [Altuğ 2010] (see also [Petersen 1996: headword 'minaret']).

For the present minaret we have n = 1 and k = 5. It may be noticed that, as in the previous example, four sides of the upper icosagon are parallel to a side of the lower square and that, contrary to it, the vertices of the square are facing vertices of the upper polygon. The reason is that in the present case k is odd, whereas it was even in example 1. The result is that the diagonals of the square induce a "natural" connection between each side of the square and five sides of the icosagon. Then, if k – 1 (here 4) points are lined up at intervals along each side of the square, this will fix five successive segments transforming it into a pseudo icosagon, each side of which being associated, by a one-to-one mapping, with the nearest side of the real icosagon to form a quadrilateral (fig. 13).

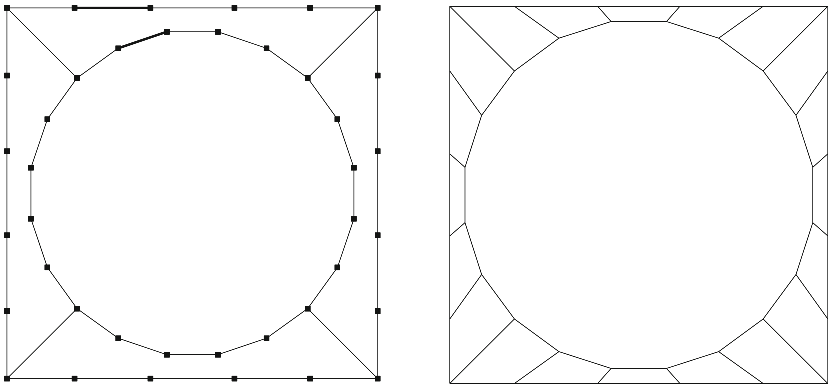

Fig. 13. Istanbul, Aghia Sofya Cami. a, left) Correspondence between the two polygons; b, right) The 20 quadrilaterals

Most of the quadrilaterals (16 out of 20) being skew, in order to get plane surfaces it will be necessary to split them into two triangles with a diagonal (fig. 14).

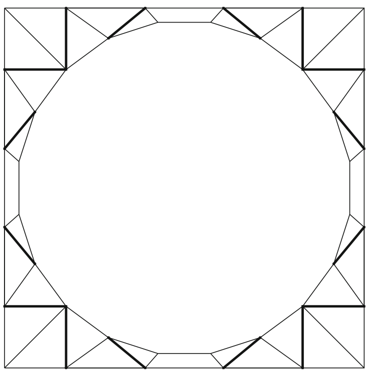

Fig. 14. Istanbul, Aghia Sofya Cami. Top view of the polyhedron

Finally, for each side of the square there will be a central trapezoid and 8 lateral triangles, four of them having their base on the square and the other four on the icosagon. Thus, on the whole there will be 4 trapezoids and 32 triangles.

Let us note that, contrary to example 1, here the trapezoids were not subdivided into triangles, doubtless because their slender shape make them similar to triangles. And the visual effect is still that of a frieze of triangles placed head to tail.

3.4. Example 3: Minaret of Aghia Sofya mosque, Iznik

Fig. 15. Iznik, Aghia Sofya Cami

In the Minaret of Aghia Sofya mosque (Iznik, fourteenth century, fig. 15) we have $n = 1$ and $k = 3$. In this case as well some sides of the dodecagon are parallel to a side of the lower square. Moreover, the vertices of the square are facing vertices of the dodecagon.

The solution adopted by the architect is basically similar to the previous one, but nevertheless it shows several differences. First the craftsman associates each vertex of the square with the vertex of the dodecagon facing it. Then, he lines up k points (i.e., 3) at intervals along each side of the square, in order to divide it into $k + 1$ consecutive sections. The segment constituted by putting together the central two sections is joined to the parallel side of the dodecagon in order to make an isosceles trapezoid (fig. 16a). This laterally determines two skew quadrilaterals which will be split into two triangles by a diagonal. To finish, contrary to the previous example but like in example 1, the trapezoids will be subdivided into three triangles, doubtless for the same reason (the division is intended to get a frieze of triangles placed head to tail) (fig. 16b). Thus a succession of 28 triangles placed head to tail is created.

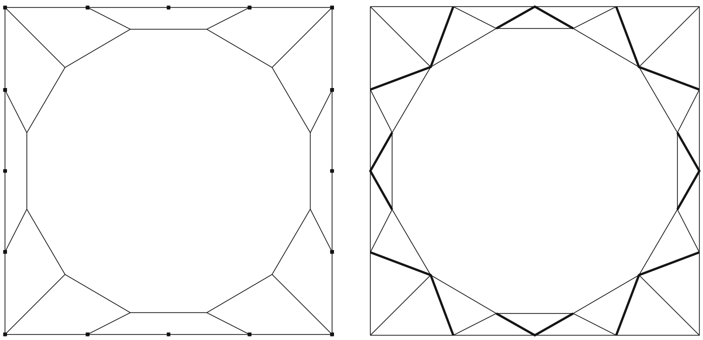

Fig. 16. Iznik, Aghia Sofya Cami. Determination of the polyhedron:
a, left) the quadrilaterals; b, right) the diagonals

In the present case this solution, already satisfying, is refined, since each of the 16 upright triangles becomes the base of a "negative" tetrahedron, created by "pushing in" a vertex under the plane of the triangle (this, so to speak, "unconvexifies" the polyhedron) (fig. 17). On the whole there is now a succession of 60 triangular sides.

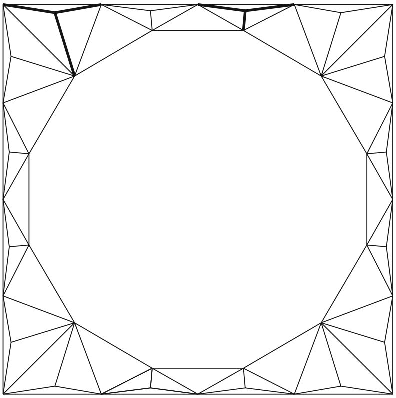

Fig. 17. Iznik, Aghia Sofya Cami. The "unconvexified" polyhedron

3.5. Example 4: West minaret of the Great Mosque of Bursa

Bursa's Great Mosque (1396, fig. 18) includes a minaret at both ends of its façade. We shall study first the West minaret.

Fig. 18. Bursa, Great Mosque (Ulu Cami). West minaret

We have here n = 2 and k = 2 as in example 1, but this time no side of the upper hexadecagon is parallel to a side of the lower octagon (thus there are no possible trapezoids). Moreover, the vertices of the octagon are facing vertices of the hexadecagon. To determine the polyhedron, the simplest solution would have been to join each vertex of the octagon with the nearest three vertices of the hexadecagon, in order to obtain 24 triangles (fig. 19a). Another solution would have been to determine k points (i.e., 2) on each side of the octagon, which would have given 40 triangles placed head to tail (fig. 19b).

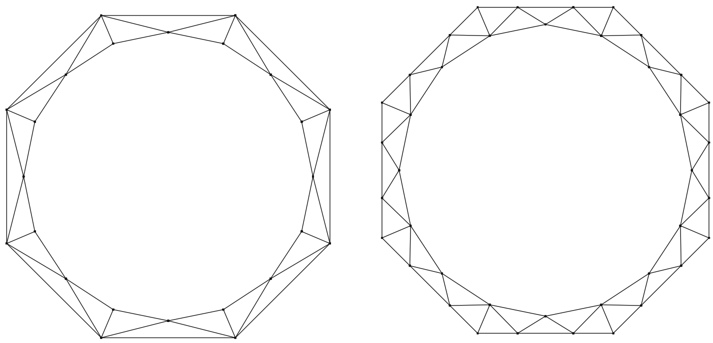

Fig. 19. Two possible connections: a, left) 24 triangles; b, right) 40 triangles

But the architect chose a third solution, certainly to get more tapering triangles (fig. 20): spread out four points on each side of the lower polygon and set a point (middle point) on each side of the upper polygon, hence 72 triangles placed head to tail.

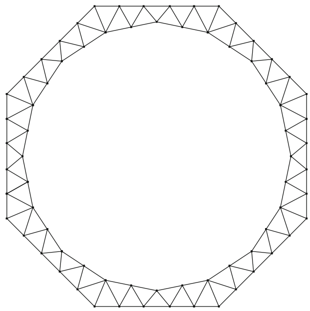

Figure 20. Bursa, Great Mosque. West minaret. The polyhedron seen from above

3.6. Example 5: East minaret of the Great Mosque of Bursa

The section of the shaft of the East minaret of the Great Mosque of Bursa (fig. 21) is hexadecagonal, as is its almost twin, but its base is square. Thus we have here $n = 1$ and $k = 4$.

Fig. 21. Bursa, Great Mosque. East minaret

As for the other minaret of this mosque, no side of the upper polygon is parallel to a side of the lower polygon. From what has been seen above, the simplest solution would have be to line up $k - 1$ points (i.e., 3) at intervals along each side of the square, in order to determine 32 triangles placed head to tail (fig. 22).

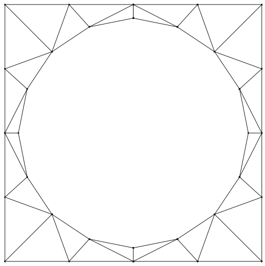

Fig. 22. A possible connection

But the triangles situated about the center of the sides of the square would then be very dissymmetric, and this is possibly the reason which led the architect to adopt another solution.

One can also see in fig. 21 that the sides of the polyhedron which are above the vertices of the square present a peculiar aspect: they are actually skew quadrilaterals made of two isosceles triangles which have the same base but are not coplanar (fig. 23).

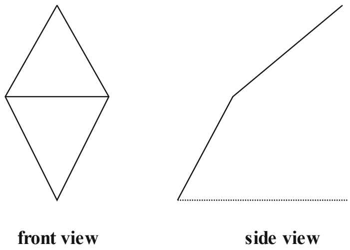

Fig. 23. Bursa, Great Mosque. The skew quadrilaterals in the corners

This skew quadrilateral is flanked on both sides by three triangles. The central part is constituted of five triangles, two of which (those with their horizontal side on the upper polygon) are "pushed in" to form a "negative" tetrahedron (cf. example 3). And finally we get a non-convex polyhedron with 68 lateral sides (including the sides of the "negative" tetrahedra) (fig. 24).

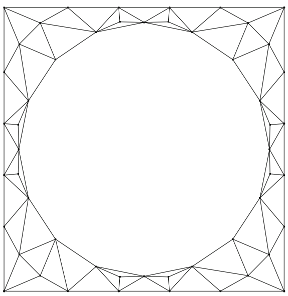

Fig. 24. Bursa, Great Mosque. East minaret. Top view

4 Conclusion

In order to synthesize the various solutions given on these examples by Ottoman architects to the question of connecting the upper part with the lower part of minarets by means of a polyhedron, it appears that seven variables can be distinguished on the whole:

- the shape of the *base* (characterised by its number of sides, $4n$);
- the shape of the *shaft* (characterised by its number of sides, $4p$);
- the integer *ratio* p/n;
- the existence (or not) of sides of the upper polygon *parallel* to sides of the lower polygon;
- the subdivision (or not) of the *trapezoids* into triangles;
- the existence (or not) of *"negative" tetrahedra* having some triangles as their bases;
- the number of *lateral sides* of the polyhedron.

This leads to Table 1, showing the great variety of the solutions found in spite of the constraints which could implicitly be observed: general structure of the minaret, number of sides of the lower polygons multiple of 4, number of sides of the upper polygon multiple of the number of sides of the lower polygon, triangles placed head to tail, and so forth.

	Example 1 Bursa Üftade	Example 2 Istanbul	Example 3 Iznik	Example 4 Bursa Ulu W.	Example 5 Bursa Ulu E.
Sides of base	8	4	4	8	4
Sides of shaft	16	20	12	16	16
Ratio	2	5	3	2	4
Parallel sides	yes	yes	yes	no	no
Subdivided trapezoids	yes	no	yes		
Negative tetrahedra	no	no	yes	no	yes
Lateral sides*	16 (32)	36	28 (60)	72	52 (68)

* For example 1, the number between brackets includes the subdivision of the trapezoids; for examples 3 and 5 it includes the sides of the "negative" tetrahedra.

Table 1. Synthesis of the studied solutions

Nevertheless, under an apparent diversity, a permanent feature can be distinguished among the minarets studied: a wish to display *visually* a frieze of triangles placed head to tail (some of them being possibly subdivided into negative tetrahedra in order to "unconvexify" the polyhedron). It resulted in a specific feature, recognizable at first sight in spite of a great variety in the arrangements of the triangles displayed between the base and the shaft. It could possibly be of interest to broaden the corpus in order to determine whether, in minarets of this type, specific values of the above variables are linked with given architects, periods or regions (throughout Ottoman empire, in Turkey and abroad[5]), but this was of course beyond the scope of this present study, which considered only the geometrical aspects of the question.

To finish with, I would just like to point out that the *pabuç* has also a functional purpose, which is to protect the minaret against violent winds and earthquakes, which occur frequently in that region. During the earthquake which shook the Izmit region of Turkey in 1999, numerous minarets collapsed or were seriously damaged. Studies undertaken subsequently [Doğangün et al. 2007] showed that the intermediate part between the shaft and the base, i.e., the *pabuç*, was the most fragile one and that Ottoman architects were apparently aware of this since, in order to reinforce this part, they had inserted iron clamps to link the blocks together; for this purpose, they used a "special technique for reinforcing and linking adjacent stone blocks with iron pieces in the vertical and horizontal directions" [Doğangün et al. 2007: 251], which proved to be effective.

Notes

1. Following Goodwin [2003], six periods can be distinguished in Ottoman architecture (1299-1922): Bursa period (1299-1437), Classical period (1437-1703), Tulip period (1703-1757), Baroque period (1757-1808), Empire period (1808-1876) and Late period (1876-1922). The minarets studied in this article belong to the Bursa and classical periods.
2. Two main types of mausoleums can be distinguished: "tower-like Seljuk graves (*türbe*) or with cupola (*kümbet*)" [Gierlichs 2008: 373, my trans.].
3. *Muqarnas*: "ornamental stalactites decorating cupolas or corbelled parts of an edifice" [Stierlin 2009: 219, my trans.].
4. When the shaft is cylindrical, the top of the intermediate part is adjusted into a convex regular polygon with $4kn$ sides, and the junction with the shaft is usually made through a torus.
5. From the mid-sixteenth to the mid-seventeenth centuries, at the height of its power, the Ottoman Empire stretched over Western Asia, Northern Africa and Southeastern Europe.

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About the author

Bernard Parzysz is Professor of mathematics – now emeritus – at the University of Orléans (France). His teaching was essentially devoted to math teachers' training and one of his research topics was – and still is – the teaching and learning of geometry. His interests include geometric mosaics, about which he published "Using Key Diagrams to Design and Construct Roman Geometric Mosaics?" in a previous issue of the *NNJ* (11, 2: 273-287).